Journal of Applied Mechanics and Technical Physics, Vol. 44, No. 2, pp. 204–211, 2003

FLOW ASYMPTOTICS OF A VISCOUS COMPRESSIBLE FLUID WITH DISCONTINUOUS INITIAL DATA

A. B. Usov

UDC 532.526:532.516

Asymptotics of a continuous solution to a plane problem on the motion of a viscous incompressible fluid with discontinuous initial velocity and pressure fields is studied by the boundary-layer method with simultaneous stretching of space and time coordinates.

Key words: *viscous compressible fluid, internal boundary layer, asymptotics.*

Introduction. The problem on a viscous compressible fluid flow with discontinuous initial data was extensively studied [1–7], but these investigations were based on various simplifying hypotheses, e.g., the absence of nonlinear convective terms in equations of viscous fluid motion was assumed [5]. The boundary-layer method [8] has not been applied to study this problem in the general statement. The boundary-layer method suggested in [9] with simultaneous stretching of space and time coordinates made it possible to construct the asymptotics of the problem solution on a very short time scale without any additional assumptions.

1. Formulation of the Problem. Let a viscous barotropic fluid occupy all the space in the absence of external forces. At the initial time, fields of perturbations of the dimensionless velocity vector ($\boldsymbol{v}$) and the density (ρ) of the fluid are specified, which are discontinuous on the convex surface Γ_0 dividing the entire fluid into the domains R_0 and Q_0 (Fig. 1):

$$R_0\colon \quad \boldsymbol{v} = \boldsymbol{v}^+, \quad \rho = \rho^+ \quad \text{for} \quad t = 0; \qquad Q_0\colon \quad \boldsymbol{v} = 0, \quad \rho = 0 \quad \text{for} \quad t = 0. \tag{1.1}$$

The functions $\boldsymbol{v}^+$ and ρ^+ are given ($\boldsymbol{v}^+|_{\Gamma_0} \neq 0$ and $\rho^+|_{\Gamma_0} > 0$).

Conditions (1.1) generate a compression wave propagating in the fluid at $t > 0$, and the velocity vector and pressure have a weak discontinuity on passing through this wave [1], i.e., there occur discontinuities in space derivatives of the components of the fluid velocity vector and in pressure. The shock front Γ_t divides the entire fluid into the domains R_t and Q_t, which are time-dependent, because the compression wave moves in the fluid (Fig. 1).

The problem is studied in the plane formulation. A moving orthogonal system of coordinates $Oy\varphi$ rigidly fitted to the curve Γ_t is introduced (O is an arbitrary point on Γ_t, y is the distance from the point to the curve Γ_t along the inner normal, and φ is the length of the arc along Γ_t counted clockwise).

Absolute isothermal motion of the fluid in the coordinate system $Oy\varphi$ in the domains R_t and Q_t in the absence of external forces is described by a system of dimensionless Navier–Stokes equations [1, 4] including the equation of continuity

$$\frac{\partial\rho}{\partial t} - c\,\frac{\partial\rho}{\partial y} + \operatorname{div}\,[(\rho + \rho_*)\boldsymbol{v}] = 0, \tag{1.2}$$

the equations of fluid motion in the coordinate system $Oy\varphi$

$$(\rho + \rho_*)\Big[\frac{\partial v_y}{\partial t} + \frac{v_\varphi}{H_1}\,\frac{\partial v_y}{\partial\varphi} + (v_y - c)\frac{\partial v_y}{\partial y} - \frac{v_\varphi^2}{H_1}\,\frac{\partial H_1}{\partial y}\Big]$$

$$= -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\Big[2\,\frac{\partial^2 v_y}{\partial y^2} + \frac{1}{H_1}\,\frac{\partial}{\partial\varphi}\Big(\frac{\partial v_\varphi}{\partial y} + \frac{1}{H_1}\,\frac{\partial v_y}{\partial\varphi} - \frac{v_\varphi}{H_1}\,\frac{\partial H_1}{\partial y}\Big) - \frac{2}{3}\,\frac{\partial\operatorname{div}\boldsymbol{v}}{\partial y}\Big], \tag{1.3}$$

Rostov State University, Rostov-on-Don 344104. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 44, No. 2, pp. 63–71, March–April, 2003. Original article submitted October 3, 2001; revision submitted April 16, 2002.

0021-8944/03/4402-0204 $25.00 © 2003 Plenum Publishing Corporation

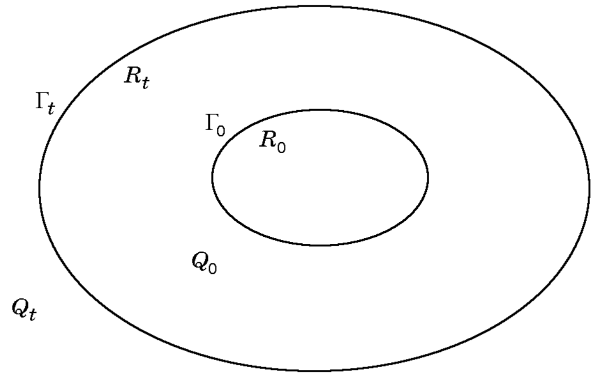

Fig. 1

$$(\rho+\rho_*)\Big[\frac{\partial v_\varphi}{\partial t}+\frac{v_\varphi}{H_1}\frac{\partial v_\varphi}{\partial \varphi}+(v_y-c)\frac{\partial v_\varphi}{\partial y}+\frac{v_\varphi(v_y-c)}{H_1}\frac{\partial H_1}{\partial y}-\frac{v_y}{H_1}\frac{\partial c}{\partial \varphi}\Big]$$
$$=-\frac{1}{H_1}\frac{\partial p}{\partial \varphi}+\frac{1}{\mathrm{Re}}\Big[\frac{2}{H_1}\frac{\partial}{\partial\varphi}\Big(\frac{1}{H_1}\frac{\partial v_\varphi}{\partial\varphi}\Big)+\frac{\partial}{\partial y}\Big(\frac{\partial v_\varphi}{\partial y}+\frac{1}{H_1}\frac{\partial v_y}{\partial\varphi}-\frac{v_\varphi}{H_1}\frac{\partial H_1}{\partial y}\Big)-\frac{2}{3H_1}\frac{\partial\,\mathrm{div}\,\boldsymbol{v}}{\partial\varphi}\Big]$$

and the equation of state

$$p=D\rho. \tag{1.4}$$

Here $\boldsymbol{v}=(v_y,v_\varphi)$ is the vector of perturbations of the dimensionless absolute velocity of the fluid particles in $Oy\varphi$, $c(\varphi,t)$ is the dimensionless displacement velocity of the curve Γ_t, p is the perturbation of dimensionless hydrodynamic pressure, ρ is the perturbation of dimensionless fluid density, ρ_* is the dimensionless fluid density in the unperturbed state, $\mathrm{Re}=\rho'v'l'/\mu'$ is the Reynolds number, ρ', v', and l' are the characteristic values of density, velocity, and length, respectively, which can be taken as the dimensional fluid density, the module of the fluid velocity vector at the initial time in the domain R_0, and the width of the domain R_0, μ' is the dynamic molecular viscosity of the fluid ($\mu'\ll 1$), $D=D'/(v')^2$, where D' is the squared of velocity of sound in the fluid, $H_1=1+y/æ(\varphi)$ is the Lamé coefficient of the coordinate system $Oy\varphi$, and $æ(\varphi)$ is the radius of curvature of Γ_t. It is assumed that the Reynolds number is independent of the Mach number [$\mathrm{M}^2=(v')^2/D'$].

Equations (1.2)–(1.4) are considered with the initial conditions (1.1), conditions at infinity, and conditions on the shock wave (for $y=0$), which moves in the fluid at a velocity $c(\varphi,t)$ [1, 8]:

$$[v_\varphi]=0,\quad [v_y]=0,\quad \Big[-p+\frac{4}{3\,\mathrm{Re}}\frac{\partial v_y}{\partial y}\Big]=0,\quad \Big[\frac{\partial v_\varphi}{\partial y}\Big]=0,\quad c=v_y. \tag{1.5}$$

Here $[f]=\lim\limits_{y\to0,\ y\in R_t} f(\varphi,y,t)-\lim\limits_{y\to0,\ y\in Q_t} f(\varphi,y,t)$ is the jump of the function f on passing the curve Γ_t.

The boundary-layer method [8] is used to construct asymptotic expansions of the solution of problem (1.1)–(1.5).

2. Constructing the Functions of the First Iteration Process. The problem solution is sought as analytic functions of the small parameter ε ($\varepsilon=1/\sqrt{\mathrm{Re}}$), i.e., in the form of the Taylor power series in ε:

$$\boldsymbol{v}=\boldsymbol{v}^{(0)}=(v_y^{(0)},v_\varphi^{(0)}),\qquad v_\varphi^{(0)}=\sum_{i=0}^{N}\varepsilon^i b_i(y,\varphi,t),\qquad v_y^{(0)}=\sum_{i=0}^{N}\varepsilon^i a_i(y,\varphi,t),$$
$$\rho=\rho^{(0)}=\sum_{i=0}^{N}\varepsilon^i\rho_i(y,\varphi,t),\qquad p=D\rho,\qquad c=c^{(0)}=\sum_{i=0}^{N}\varepsilon^i c_i^{(0)}(\varphi,t). \tag{2.1}$$

The functions a_i, b_i, $c_i^{(0)}$, and ρ_i ($i=0,1,\ldots,N$) are called the functions of the first iteration process. By substituting series (2.1) into Eqs. (1.2)–(1.4), we obtain Euler equations of an ideal barotropic fluid for determining the functions of the first iteration process. For an ideal compressible fluid, the boundary conditions at the shock wave under the initial conditions (1.1) have the form [1]

$$[(\rho+\rho_*)(v_y-c)]=0,\qquad [(\rho+\rho_*)v_y(v_y-c)+p]=0,\qquad [v_\varphi]=0 \tag{2.2}$$

and cannot satisfy all the conditions (1.5). It is assumed that boundary conditions (2.2) and initial conditions (1.1) are consistent, i.e., conditions (2.2) are fulfilled for $t = 0$. In the opposite case, the initial discontinuity in the ideal fluid decays. Therefore, at $y = 0$, the following equalities hold true:

$$c(0,\varphi) = v_y^+(0,\varphi)(\rho^+(0,\varphi) + \rho_*)/\rho^+(0,\varphi),$$
$$(\rho^+(0,\varphi) + \rho_*)v_y^+(0,\varphi)(v_y^+(0,\varphi) - c(0,\varphi)) + D\rho^+(0,\varphi) = 0, \qquad v_\varphi^+(0,\varphi) = 0. \tag{2.3}$$

Here $\boldsymbol{v}^+ = (v_y^+, v_\varphi^+)$.

Seeking the solution of problem (1.1)–(1.5) as series (2.1), it is impossible to satisfy all the conditions (1.5). Near the surface Γ_t, a thin fluid layer is formed (internal boundary layer [4, 5]) where a drastic change in fluid velocity and pressure is observed. The processes in the boundary layer are described by the boundary-layer functions.

3. Construction of the Boundary-Layer Functions. The solution of problem (1.1)–(1.5) is sought in the form [8]

$$\boldsymbol{v} = \boldsymbol{v}^{(0)} + \boldsymbol{v}^{(1)} + \boldsymbol{v}^{(2)}, \qquad \rho = \rho^{(0)} + \rho^{(1)} + \rho^{(2)},$$
$$p = p^{(0)} + p^{(1)} + p^{(2)}, \qquad c = c^{(0)} + c^{(1)}, \qquad p^{(k)} = D\rho^{(k)}, \quad k = 0, 1, 2,$$
$$v_y^{(1)} = \sum_{i=N_1}^{N} \varepsilon^i h_i^{(1)}(s,\varphi,\tau), \qquad v_y^{(2)} = \sum_{i=N_1}^{N} \varepsilon^i h_i^{(2)}(s,\varphi,\tau), \tag{3.1}$$
$$v_\varphi^{(1)} = \sum_{i=N_2}^{N} \varepsilon^i g_i^{(1)}(s,\varphi,\tau), \qquad v_\varphi^{(2)} = \sum_{i=N_2}^{N} \varepsilon^i g_i^{(2)}(s,\varphi,\tau),$$
$$\rho^{(1)} = \sum_{i=N_3}^{N} \varepsilon^i \alpha_i^{(1)}(s,\varphi,\tau), \qquad \rho^{(2)} = \sum_{i=N_3}^{N} \varepsilon^i \alpha_i^{(2)}(s,\varphi,\tau), \qquad c^{(1)} = \sum_{i=N_4}^{N} \varepsilon^i c_i^{(1)}(\varphi,\tau).$$

The constants $N_1, \ldots, N_4$ are found in the process of constructing the boundary-layer functions. The functions $\boldsymbol{v}^{(0)}$, $\rho^{(0)}$, $p^{(0)}$, and $c^{(0)}$ are the functions of the first iteration process and have a strong discontinuity at the surface Γ_t. The functions $g_i^{(m)}$, $h_i^{(m)}$, $c_i^{(1)}$, and $\alpha_i^{(m)}$ $(m = 1, 2)$ are called the functions of the second iteration process (functions of the internal boundary layer in a viscous fluid). We assume that the boundary-layer functions, unlike [4–8], depend on two stretched variables $s = y/\varepsilon^{k_1}$ and $\tau = t/\varepsilon^{k_2}$ $(k_1, k_2 = \text{const})$. The boundary-layer functions are determined in the vicinity of the curve Γ_t on its opposite sides (in the regions R_t and Q_t) and secure the continuity of the viscous fluid flow on passing through Γ_t.

By reasoning similar to [9] and discarding the requirement to conserve the divergence of the velocity vector in every approximation, we obtain the following variants of boundary-layer stretching:

$$k_1 = 1 + 0.5k_2, \quad k_2 > 2, \quad N_1 = 0, \quad N_2 = k_2, \quad N_3 = -k_1 + k_2, \quad N_4 = N_1; \tag{3.2}$$

$$k_1 = 2, \quad k_2 = 2, \quad N_1 = 0, \quad N_2 = 2, \quad N_3 = 0, \quad N_4 = 0. \tag{3.3}$$

The solution of problem (1.1)–(1.5) is unique [1]; therefore, the parameters of boundary-layer stretching specified by formulas (3.2) and (3.3) lead to different representations of the same asymptotics of the problem solution. The equivalence of these representations on short time scales is proved in a similar way [9]. It is impossible to construct functions of the internal boundary layer by the boundary-layer method [8] without stretching the time ($k_2 = 0$) and without making additional assumptions. For $k_2 \neq 0$, the internal boundary layer can be constructed without any additional assumptions.

Let us construct a boundary layer for the case

$$k_1 = 3, \quad k_2 = 4, \quad N_1 = 0, \quad N_2 = 4, \quad N_3 = 1, \quad N_4 = 1.$$

The solution of problem (1.1)–(1.5) is sought in the form of series (3.1). In an ideal fluid, there is a shock-wave front producing a strong discontinuity; in a viscous fluid, this discontinuity is smoothed by dissipation forces. We substitute series (3.1) into Eqs. (1.2) and (1.3) and take into account the results of the first iteration process: the functions of this process satisfy homogeneous (in the basic approximation) and inhomogeneous (in the subsequent approximations) Euler equations of an ideal compressible fluid. In the resulting equations, we expand the functions depending on the "slow" time $t = \tau\varepsilon^4$ and the variable $y = s\varepsilon^3$ in terms of the Taylor series at the points $t = 0$ and $y = 0$. Arranging coefficients of similar powers of ε, we obtain the boundary-layer equations.

In the basic approximation, the boundary-layer equations have the form

$$\frac{\partial g_4^{(1)}}{\partial\tau} - h_0^{(1)}\frac{\partial c_0^{(0)}}{\partial\varphi}\Big|_{t=0} + (h_0^{(1)} - c_0^{(1)})\frac{\partial v_\varphi^+}{\partial y}\Big|_{y=0} - (h_0^{(1)} + v_y^+\Big|_{y=0})\frac{\partial c_0^{(1)}}{\partial\varphi} = \frac{1}{\rho_* + \rho^+|_{y=0}}\frac{\partial^2 g_4^{(1)}}{\partial s^2},$$

$$\frac{\partial g_4^{(2)}}{\partial\tau} - h_0^{(2)}\Big[\frac{\partial c_0^{(0)}}{\partial\varphi}\Big|_{t=0} + \frac{\partial c_0^{(1)}}{\partial\varphi}\Big] = \frac{1}{\rho_*}\frac{\partial^2 g_4^{(2)}}{\partial s^2},$$

$$\frac{\partial h_0^{(1)}}{\partial\tau} = \frac{4}{3(\rho_* + \rho^+|_{y=0})}\frac{\partial^2 h_0^{(1)}}{\partial s^2}, \qquad \frac{\partial h_0^{(2)}}{\partial\tau} = \frac{4}{3\rho_*}\frac{\partial^2 h_0^{(2)}}{\partial s^2}, \tag{3.4}$$

$$\frac{\partial\alpha_1^{(1)}}{\partial\tau} + (\rho^+\Big|_{y=0} + \rho_*)\frac{\partial h_0^{(1)}}{\partial s} = 0, \qquad \frac{\partial\alpha_1^{(2)}}{\partial\tau} + \rho_*\frac{\partial h_0^{(2)}}{\partial s} = 0,$$

and in the subsequent approximations, they become ($k = 5, 6, \ldots, N$)

$$\frac{\partial g_k^{(1)}}{\partial\tau} - h_{k-4}^{(1)}\frac{\partial c_0^{(0)}}{\partial\varphi}\Big|_{t=0} + (h_{k-4}^{(1)} - c_{k-4}^{(1)})\frac{\partial v_\varphi^+}{\partial y}\Big|_{y=0} - h_0^{(1)}\frac{\partial c_{k-4}^{(1)}}{\partial\varphi} - h_{k-4}^{(1)}\frac{\partial c_0^{(1)}}{\partial\varphi}$$

$$= \frac{1}{\rho_* + \rho^+|_{y=0}}\frac{\partial^2 g_k^{(1)}}{\partial s^2} + \Phi_k^{(1)},$$

$$\frac{\partial g_k^{(2)}}{\partial\tau} - h_{k-4}^{(2)}\Big[\frac{\partial c_0^{(0)}}{\partial\varphi}\Big|_{t=0} + \frac{\partial c_0^{(1)}}{\partial\varphi}\Big] - h_0^{(2)}\frac{\partial c_{k-4}^{(1)}}{\partial\varphi} = \frac{1}{\rho_*}\frac{\partial^2 g_k^{(2)}}{\partial s^2} + \Phi_k^{(2)},$$

$$\frac{\partial h_{k-4}^{(1)}}{\partial\tau} = \frac{4}{3(\rho_* + \rho^+|_{y=0})}\frac{\partial^2 h_{k-4}^{(1)}}{\partial s^2} + \Phi_k^{(3)}, \qquad \frac{\partial h_{k-4}^{(2)}}{\partial\tau} = \frac{4}{3\rho_*}\frac{\partial^2 h_{k-4}^{(2)}}{\partial s^2} + \Phi_k^{(4)}, \tag{3.5}$$

$$\frac{\partial\alpha_{k-3}^{(1)}}{\partial\tau} + (\rho^+|_{y=0} + \rho_*)\frac{\partial h_{k-4}^{(1)}}{\partial s} = \Phi_k^{(5)}, \qquad \frac{\partial\alpha_{k-3}^{(2)}}{\partial\tau} + \rho_*\frac{\partial h_{k-4}^{(2)}}{\partial s} = \Phi_k^{(6)},$$

where $\Phi_k^{(j)}$ ($j = 1, 2, \ldots, 6$) are the known functions depending on the functions of the boundary layer and of the first iteration process calculated as a result of problem solution for $i = 4, 5, \ldots, k-1$. By virtue of (2.3), the function $c_0^{(0)}(0, \varphi)$ has the form

$$c_0^{(0)}(0, \varphi) = v_y^+(0, \varphi)(\rho^+(0, \varphi) + \rho_*)/\rho^+(0, \varphi).$$

Therefore, problems for the boundary-layer functions in the first and subsequent approximations [see (3.5)] differ from the problem in the basic approximation [see (3.4)] only by the presence of certain heterogeneity.

In deriving boundary conditions on the curve Γ_t for the functions of the first and second iteration processes, we take into account that the boundary conditions on Γ_t for the case of an ideal compressible fluid have the form (2.2). By substituting series (2.1) into equalities (2.2) and arranging coefficients of similar powers of ε, we obtain the boundary conditions on Γ_t for the functions of the first iteration process at $y = 0$:

$$\text{for } \varepsilon^0, \qquad \begin{aligned} &[D\rho_0 + a_0(\rho_0 + \rho_*)(a_0 - c_0^{(0)})] = 0, \quad [b_0] = 0, \\ &(\rho_0 + \rho_*)(a_0 - c_0^{(0)}) = -\rho_* c_0^{(0)}; \end{aligned} \tag{3.6}$$

$$\text{for } \varepsilon^k, \qquad \begin{aligned} &\Big[D\rho_k + \sum_{i+j+m=k} a_j\rho_j(a_m - c_m^{(0)}) + \sum_{j+m=k} a_j\rho_*(a_m - c_m^{(0)})\Big] = 0, \\ &[b_k] = 0, \quad \sum_{j+i=k}\rho_i(a_j - c_j^{(0)}) = -\rho_* c_k^{(0)}, \quad k = 1, 2, \ldots, N. \end{aligned} \tag{3.7}$$

We substitute series (3.1) into conditions (1.5). By expanding the functions depending on the "slow" time t in terms of the Taylor series at the point $t = 0$ and arranging coefficients of similar powers ε, we find the boundary conditions for the functions of the second iteration process at $s = 0$:

$$\frac{\partial g_4^{(1)}}{\partial s} = -\frac{\partial g_4^{(2)}}{\partial s}, \qquad \frac{\partial h_0^{(1)}}{\partial s} = -\frac{\partial h_0^{(2)}}{\partial s}, \qquad g_4^{(1)} = g_4^{(2)},$$

$$h_0^{(1)} + v_y^+\Big|_{y=0} = h_0^{(2)}, \qquad c_0^{(1)} = h_0^{(1)} - v_y^+\Big|_{y=0}\rho_*/(\rho^+|_{y=0}); \tag{3.8}$$

$$\frac{\partial g_k^{(1)}}{\partial s} = -\frac{\partial g_k^{(2)}}{\partial s}, \qquad \frac{\partial h_{k-4}^{(1)}}{\partial s} = -\frac{\partial h_{k-4}^{(2)}}{\partial s}, \qquad g_k^{(1)} = g_k^{(2)}, \quad k = 5, 6, \dots, N,$$

$$h_{k-4}^{(1)} + \sum_{i+4m=k-4} \frac{\tau^m}{m!} \frac{\partial^m a_i^{(1)}}{\partial t^m}\Big|_{t=0} = h_{k-4}^{(2)} + \sum_{i+4m=k-4} \frac{\tau^m}{m!} \frac{\partial^m a_i^{(2)}}{\partial t^m}\Big|_{t=0}, \tag{3.9}$$

$$c_{k-4}^{(1)} = h_{k-4}^{(1)} + \sum_{i+4m=k-4} \frac{\tau^m}{m!} \frac{\partial^m a_i^{(1)}}{\partial t^m}\Big|_{t=0} - \sum_{i+4m=k-4} \frac{\tau^m}{m!} \frac{\partial^m c_i^{(0)}}{\partial t^m}\Big|_{t=0}.$$

Here $a_i^{(1)} = \lim\limits_{y\to 0,\ y\in R_t} a_i(\varphi, y, t)$ and $a_i^{(2)} = \lim\limits_{y\to 0,\ y\in Q_t} a_i(\varphi, y, t)$. In formulas (3.8) and (3.9), the functions of the first iteration process are known from the solutions of the respective problems.

The algorithm of solving the problem is as follows:

1. The functions a_k, b_k, $c_k^{(0)}$, ρ_k, and p_k $(k = 0)$ are found from homogeneous (in the basic approximation) and inhomogeneous (in the subsequent approximations) Euler equations of an ideal compressible fluid with conditions (3.6) or (3.7), conditions at infinity, and the following initial conditions: $\boldsymbol{v}_k^{(0)} = 0$ and $\rho_k = 0$ $(k = 1, 2, \dots, N)$, $\boldsymbol{v}_0^{(0)} = \boldsymbol{v}^+$ and $\rho_0 = \rho^+$ in the region R_0, and $\boldsymbol{v}_0^{(0)} = 0$ and $\rho_0 = 0$ at $t = 0$ in the region Q_0.

2. Problem (3.5) is solved with decreasing conditions at infinity, zero initial conditions, and conditions (3.9). The functions $g_{k+4}^{(m)}$, $h_k^{(m)}$, $c_k^{(1)}$, and $\alpha_{k+1}^{(m)}$ $(m = 1,\ 2$ and $k = 0)$ are determined.

3. The procedure is repeated for the subsequent values of k.

The leading terms of the boundary-layer corrections to the solution i.e., the solution of Eqs. (3.4) with conditions (3.8), zero initial conditions, and decreasing conditions at infinity, are written as [10]

$$g_4^{(1)}(s, \varphi, \tau) = \int_0^\tau \int_0^\infty \Big[- h_0^{(1)}(\eta, \varphi, t)\, \frac{\partial c_0^{(0)}(\varphi, 0)}{\partial \varphi}$$

$$+ (h_0^{(1)}(\eta, \varphi, t) - c_0^{(1)}(\varphi, t)) \frac{\partial v_\varphi^+}{\partial y}(0, \varphi) - (h_0^{(1)}(\eta, \varphi, t) + v_y^+(0, \varphi)) \frac{\partial c_0^{(1)}(\varphi, t)}{\partial \varphi} \Big]$$

$$\times \Big[K\Big(s - \eta, \frac{\tau - t}{\rho^+(0, \varphi) + \rho_*}\Big) + K\Big(s + \eta, \frac{\tau - t}{\rho^+(0, \varphi) + \rho_*}\Big) \Big]\, d\eta\, dt$$

$$- \frac{2}{\rho_* + \rho^+(0, \varphi)} \int_0^\tau \Big[\frac{\partial g_4^{(2)}}{\partial s}(0, \varphi, t) K\Big(s, \frac{\tau - t}{\rho_* + \rho^+(0, \varphi)}\Big) \Big]\, dt, \tag{3.10}$$

$$g_4^{(2)}(s, \varphi, \tau) = \int_0^\tau \int_0^\infty \Big[h_0^{(2)}(\eta, \varphi, t) \Big(\frac{\partial c_0^{(0)}(\varphi, t)}{\partial \varphi} + \frac{\partial c_0^{(1)}(\varphi, t)}{\partial \varphi} \Big) \Big]$$

$$\times \Big[K\Big(s - \eta, \frac{\tau - t}{\rho_*}\Big) + K\Big(s + \eta, \frac{\tau - t}{\rho_*}\Big) \Big]\, d\eta\, dt + \frac{2}{\rho_*} \int_0^\tau \Big[g_4^{(1)}(0, \varphi, t)\, \frac{\partial K(s - \eta, (\tau - t)/\rho_*)}{\partial \eta}\Big|_{\eta=0} \Big]\, dt;$$

$$h_0^{(1)} = \frac{-8}{3(\rho_* + \rho^+(0, \varphi))} \int_0^\tau \Big[\frac{\partial h_0^{(2)}}{\partial s}(0, \varphi, t) K\Big(s, \frac{4(\tau - t)}{3(\rho_* + \rho^+(0, \varphi))}\Big) \Big]\, dt, \tag{3.11}$$

$$h_0^{(2)} = \frac{8}{3\rho_*} \int\limits_0^\tau \Big[\frac{\partial K(s-\eta, 4(\tau-t)/(3\rho_*))}{\partial \eta} \Big|_{\eta=0} (h_0^{(1)}(0,\varphi,t) + v_y^+(0,\varphi)) \Big]\, dt;$$

$$\alpha_1^{(1)}(s,\varphi,\tau) = -(\rho^+(0,\varphi) + \rho_*) \int\limits_0^\tau \frac{\partial h_0^{(1)}(s,\varphi,t)}{\partial s}\, dt,$$

$$\alpha_1^{(2)}(s,\varphi,\tau) = -\rho_* \int\limits_0^\tau \frac{\partial h_0^{(2)}(s,\varphi,t)}{\partial s}\, dt, \tag{3.12}$$

$$c_0^{(1)}(\varphi,\tau) = h_0^{(1)}(0,\varphi,\tau) + v_y^+(0,\varphi) - v_y^+(0,\varphi)(\rho^+(0,\varphi) + \rho_*)/\rho^+(0,\varphi),$$

where $K(s,t) = \exp(-s^2/(4t))/\sqrt{4\pi t}$.

The systems of two integral equations (3.10) and (3.11) are solved by the method of successive approximations [11]. Let us prove that the integral operator $\boldsymbol{I} = (I_1, I_2)$, where

$$I_1 f_1 = \frac{8}{3(\rho_* + \rho^+|_{y=0})} \int\limits_0^\tau \Big[K\Big(s, \frac{4(\tau-t)}{3(\rho_* + \rho^+|_{y=0})}\Big) f_1(0,\varphi,t) \Big]\, dt,$$

$$I_2 f_2 = \frac{8}{3\rho_*} \int\limits_0^\tau \Big[\frac{\partial K(s-\eta, 4(\tau-t)/(3\rho_*))}{\partial \eta} \Big|_{\eta=0} f_2(0,\varphi,t) \Big]\, dt,$$

is contractive in the space of continuous vector functions $\boldsymbol{f} = (f_1(s,\varphi,\tau),\ f_2(s,\varphi,\tau))$. For definiteness, as a norm of the operator $\boldsymbol{I}$, we take

$$\|\boldsymbol{I}\| = \max(\|I_1\|, \|I_2\|), \qquad \|I_k\| = \sup_{\|f_k\| \leqslant 1} |I_k f_k|, \qquad k = 1,2.$$

By using the change of variables in the integrand, we obtain

$$\|I_1\| \leqslant \frac{s}{\sqrt{\pi}} \int\limits_{z_1}^\infty \frac{\exp(-r^2)}{r^2}\, dr = \frac{1}{\sqrt{\pi}} J_1, \quad \|I_2\| \leqslant \frac{2}{\sqrt{\pi}} \int\limits_{z_2}^\infty \exp(-r^2)\, dr = \frac{1}{\sqrt{\pi}} J_2,$$

where

$$z_1 = s\sqrt{3(\rho_* + \rho^+|_{y=0})/(16\tau)}; \qquad z_2 = s\sqrt{3\rho_*/(16\tau)};$$

$$J_1 = s \int\limits_{z_1}^\infty \frac{\exp(-r^2)}{r^2}\, dr; \qquad J_2 = 2 \int\limits_{z_2}^\infty \exp(-r^2)\, dr.$$

For every $\varepsilon > 0$, there exists a number $C > 0$ such that $J_1 < \varepsilon$ and $J_2 < \varepsilon$ if $s\sqrt{3\rho_*/(16\tau)} > C$. In addition, the operator $\boldsymbol{I}$ does not lead out continuous vector functions from the space of continuous vector functions. Consequently, the operator $\boldsymbol{I}$ is contractive in the space of continuous vector functions for $s\sqrt{3\rho_*/(16\tau)} > C^*$, where C^* is constructed on the basis of $\varepsilon^* = \sqrt{\pi}/2$.

The systems of integral equations (3.10) and (3.11) are solved by the method of successive approximations, which, according to the principle of contractive mappings, converges.

In the case where the curve Γ_t is a circle of a certain radius, $v_y^+ = \text{const}$, $v_\varphi^+ = 0$, $\rho^+ = \text{const}$, and conditions (2.3) are fulfilled for $t = 0$, we obtain

$$v_\varphi^{(0)}(y,\varphi,t) = 0, \qquad v_y^{(0)}(y,\varphi,t) = v_y^+, \qquad \rho^{(0)}(y,\varphi,t) = \rho^+,$$

and asymptotics of (3.1) are written in the following form (as $\varepsilon \to 0$):

$$v_\varphi = 0, \quad v_y = v_y^+ + h_0^{(1)} + h_0^{(2)} + O(\varepsilon), \quad \rho = \rho^+ + \varepsilon(\alpha_1^{(1)} + \alpha_1^{(2)}) + O(\varepsilon^2),$$

$$c = v_y^+(0,\varphi) + h_0^{(1)}(0,\varphi,\tau) + O(\varepsilon) = h_0^{(2)}(0,\varphi,\tau) + O(\varepsilon).$$

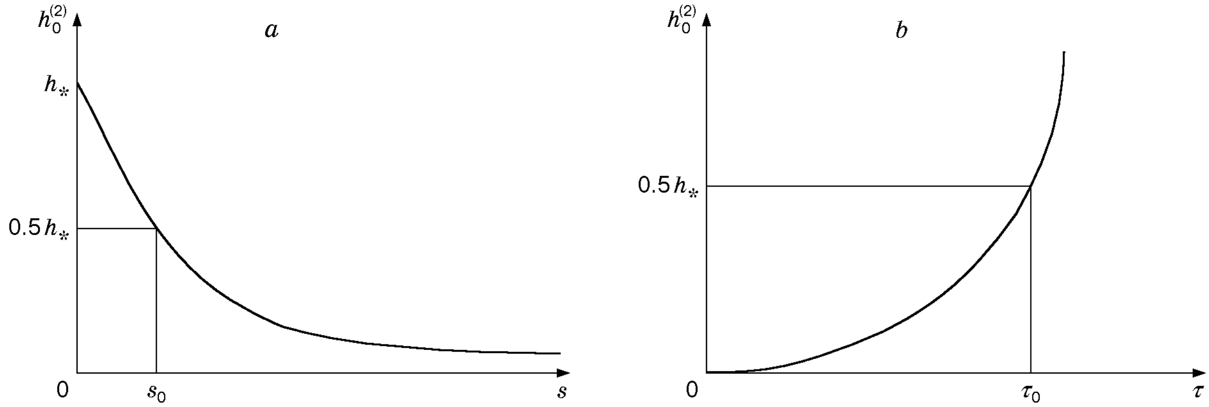

Fig. 2

At the same time, in the basic approximation of the method of successive approximations, we have $[z = s\sqrt{3\rho_*/(16\pi\tau)} = y\sqrt{3\rho_*/(16\pi t)}\,/\varepsilon]$

$$h_0^{(2)} = 2v_y^+\Big|_{y=0}\int\limits_z^\infty \exp(-x^2)\,dx,$$

$$h_0^{(1)} = \frac{64}{3(\rho_* + \rho^+|_{y=0})}\,\frac{\sqrt{\pi}}{\sqrt{3\rho_*}}\,v_y^+\Big|_{y=0}\int\limits_0^\tau \sqrt{t}K\Big(s, \frac{4(\tau - t)}{3(\rho_* + \rho^+|_{y=0})}\Big)\,dt.$$

The functions $\alpha_1^{(1)}$ and $\alpha_1^{(2)}$ are obtained by formulas (3.12). The compression-wave velocity is finite and, in the basic approximation, independent of time:

$$c = \sqrt{\pi}v_y^+\Big|_{y=0} + O(\varepsilon).$$

The function $h_0^{(2)}$ is expressed in terms of an additional probability integral [10]. Figure 2a and b shows the functions $h_0^{(2)}(s)$ and $h_0^{(2)}(\tau)$ at the fixed points $\tau = \tau_*$ and $s = s_*$, respectively. Here $h_* = \sqrt{\pi}v_y^+\Big|_{y=0}$, $s_0 = 1.92\sqrt{\pi\tau_*}/\sqrt{3\rho_*}$, and $\tau_0 = s_*^2 3\rho_*/(4\pi)$. It follows from Fig. 2, for example, that $s_0 = 19.631$ ($y_0 = s_0\varepsilon^3 = 1.96\cdot 10^{-5}$) for $\rho_* = 1$, $\varepsilon = 0.01$, and $t_* = 1$ ($\tau_* = t_*/\varepsilon^4 = 10^8$). For $\rho_* = 1$, $\varepsilon = 0.01$, and $y_* = 0.005$ ($s_* = 5000$), we have $\tau_0 = 5.78\cdot 10^6$ ($t_0 = 0.0578$). The curves in Fig. 2 allow us to evaluate the order of the internal boundary-layer thickness [4] and its time evolution.

Conclusions. Asymptotic expansions (with vanishing viscosity) of the problem solution are constructed by using simultaneous stretching of space and time coordinates in the internal boundary layer of a viscous compressible fluid. The internal boundary layer is constructed without any additional assumptions of the fluid-flow character. The leading terms of asymptotic expansions of the problem solution are derived from the system of integral equations solved by the method of successive approximations. The boundary-layer corrections of higher orders of smallness at vanishing viscosity are found from a similar system of equations, which is also solved by the method of successive approximations. Therefore, the suggested modification of the boundary-layer method enables us to construct a generic algorithm for calculating the functions of the internal boundary layer in a viscous compressible fluid with any desired degree of accuracy.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 01-01-00105).

REFERENCES

1. N. E. Kochin, I. A. Kibel', and N. V. Rose, *Theoretical Hydromechanics* [in Russian], Vol. 1, Fizmatgiz, Moscow (1963).
2. A. G. Kulikovskii and G. A. Lubimov, *Magnetic Hydrodynamics* [in Russian], Fizmatgiz, Moscow (1962).

3. R. Courant and K. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience, New York (1948).
4. L. G. Loitsyanskii, *Laminar Boundary Layer* [in Russian], Fizmatgiz, Moscow (1962).
5. L. S. Srubshchik and V. I. Yudovich, "Asymptotics of weak discontinuities of fluid flow at vanishing viscosity," *Dokl. Akad. Nauk SSSR*, **199**, No. 3, 563– 566 (1971).
6. G. G. Chernyi, "Laminar motions of gas and fluid in the boundary layer with a discontinuity surface," *Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk*, No. 2, 38–68 (1954).
7. V. I. Yudovich, "On asymptotic integration of the Navier–Stokes system at a shock applied to a fluid," Rostov-on-Don (1993). Deposited at VINITI, 07.14.93, No. 1981-B93.
8. M. I. Vishik and L. A. Lusternik, "Regular degeneracy and boundary layer for linear differential equations with a small parameter," *Usp. Mat. Nauk*, **12**, No. 5, 3–120 (1957).
9. A. B. Usov, "Asymptotic behavior of a flow with a free surface," *J. Appl. Mech. Tech. Phys.*, **37**, No. 1, 41–47 (1996).
10. A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* [in Russian], Gostekhteoretizdat, Moscow (1953).
11. G. I. Marchuk, *Methods of Computational Mathematics* [in Russian], Nauka, Moscow (1989).